Mid-Semestral Exam
Algebra-IV
B. Math - Second year
2014-2015

Time: 3 hrs
Max score: 100

Answer all questions.

(1) State true or false. Justify your answers. No marks will be awarded in the absence of proper justification.
(a) For fields $F \subseteq K$, and $\alpha \in K$, if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.
(b) The regular 5-gon is not constructible by straightedge and compass.
(c) If $F \subseteq E \subseteq K$ are fields, such that $K|E$ and $E|F$ are both Galois extensions, then $K|F$ is also a Galois extension.
(d) A polynomial over a field of characteristic zero is separable if and only if it is the product of distinct irreducible polynomials.
(e) If K is a finite field of characteristic p, then every element of K has a unique p-th root in K. (6×5)

(2) (a) Show that if F is a field with $char(F) \neq 2$, and if K is a quadratic extension of F, then $K = F(\sqrt{d})$ for some $d \in F$, d not a square in F.
(b) Find all quadratic extensions of $\mathbb{Q}$ which contain a primitive pth root of unity ζ for some prime $p \neq 2$. (6+4)

(3) Prove that there exists finite fields of order p^n for any prime p and any integer $n \geq 1$, and are unique up to isomorphism. (8)

(4) (a) Let $f(x) \in F[x]$ be a polynomial of degree n. Let K be its splitting field. Show that $[K : F]$ divides $n!$.
(b) Describe the splitting field of the polynomial $x^5 - 7$ over $\mathbb{Q}$, and find the degree of the splitting field over $\mathbb{Q}$. (8+6)

(5) (a) Let n be an odd integer such that F contains a primitive n-th root of unity and $char(F) \neq 2$. Show that F also contains a primitive $2n$-th root of unity.

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(b) Let K be a finite extension of $\mathbb{Q}$. Show that there is only a finite number of roots of unity in K. (6+8)

(6) Prove that the extension $K|F$ is Galois if and only if K is the splitting field of some separable polynomial over F. (10)

(7) Consider the polynomial $f(x) = x^4 - 3x^2 - 10 \in \mathbb{Q}[x]$.
(a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
(b) Describe the Galois group G of the extension $K|\mathbb{Q}$.
(c) Show the correspondence between all the subgroups of G and all the subfields of K containing $\mathbb{Q}$. (3+5+6)